

## Comments on bounds on central charges in $\mathcal{N} = 1$ superconformal theories

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# Comments on bounds on central charges in $\mathcal{N} = 1$ superconformal theories

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**ABSTRACT:** The ratio of central charges in four-dimensional *CFTs* has been suggested by Hofman and Maldacena to lie within an interval whose boundaries are fixed by the number of supersymmetries. We compute this ratio for a set of interacting  $\mathcal{N} = 1$  superconformal field theories which arise as *RG* fixed points of supersymmetric Yang-Mills theories with adjoint and fundamental matter. We do not find violations of the proposed bounds, which appear to be saturated by free field theories.

**KEYWORDS:** Supersymmetric gauge theory, Duality in Gauge Field Theories, Anomalies in Field and String Theories

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**Contents**

<b>1</b>	<b>Introduction and summary</b>	<b>1</b>
<b>2</b>	<b><i>sQCD</i> with one adjoint</b>	<b>3</b>
2.1	Vanishing superpotential	3
2.2	Adjoint <i>sQCD</i> with polynomial superpotential	4
2.3	Strong-weak coupling duality	6
<b>3</b>	<b><i>sQCD</i> with two adjoints</b>	<b>8</b>
3.1	$\hat{D}$	9
3.2	$D_{k+2}$	10
3.3	$\hat{E}$	11
3.4	$E_6$	12
3.5	$E_7$	13
3.6	$E_8$	14
<b>4</b>	<b>Conclusions</b>	<b>14</b>

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**1 Introduction and summary**

Weyl anomaly in four-dimensional conformal field theories contains a term proportional to the Euler density, and a square of the Weyl tensor. The properly normalized coefficients in front of these terms are the  $a$  and  $c$  central charges of the conformal group. Their values can be determined from the three and two-point functions of stress-energy tensor. In the  $\mathcal{N} = 1$  superconformal theories one can relate  $a$  and  $c$  to the 't Hooft anomalies involving the  $R$ -current [1, 2]. Hence, the values of  $a$  and  $c$  can be computed through

$$a = \frac{3}{32} (3Tr R^3 - Tr R), \quad c = \frac{1}{32} (9Tr R^3 - 5Tr R), \quad (1.1)$$

provided one can identify the correct  $U(1)_R$  symmetry in the  $IR$ . The problem is that any global  $U(1)$  which commutes with the superconformal group can become part of the  $U(1)_R$  symmetry in the  $IR$ . The solution, worked out by Intriligator and Wecht [3], is that the correct  $U(1)_R$  symmetry maximizes  $a$ . If the  $IR$  fixed point is not too strongly coupled the maximization of  $a$  allows computing the values of the central charges in a variety of interesting examples. Otherwise, the appearance of “accidental symmetries” which are not visible in the UV would complicate the situation.

The central charges  $a$  and  $c$  count the degrees of freedom of the  $CFT$ . It is therefore natural to ask whether they satisfy any nontrivial constraints. While positivity of  $c$  follows from the unitarity of the theory [1, 2], other properties have more conjectural nature. For

example,  $a$  has been proposed to satisfy an analog of Zamolodchikov  $c$ -theorem [4] in four dimensions [5]. (See [2, 6–9] for some work in this direction and [10] which argued that a counterexample exists.)

Recently new bounds on the ratio of central charges in  $\mathcal{N} = 1$  superconformal theories have been suggested in [11].<sup>1</sup> By requiring the positivity of the energy flux, [11] derived the following inequality for the  $\mathcal{N} = 1$  superconformal theories,

$$-\frac{1}{2} \leq \alpha \leq \frac{1}{2}, \tag{1.2}$$

where we define

$$\alpha = \frac{a}{c} - 1. \tag{1.3}$$

The bound (1.2) is especially interesting in light of the recent works [13] (see also [14]) where it has been related to the causality bound in Gauss-Bonnet gravitational theory. There are similar constraints for the  $CFT$ s with  $\mathcal{N} = 0, 2$  supersymmetries, with the  $\mathcal{N} = 2$  bound subsequently proven in [15]. Assuming positivity of the energy flux seems natural and can even be proven for free field theories. However, there is no proof for a generic interacting  $CFT$ .

In this note we compute the values of  $\alpha$ , as defined in (1.3), for a set of interacting  $CFT$ s studied in [16]. This includes the IR fixed points of supersymmetric  $QCD$  with one and two adjoints with all possible relevant superpotential deformations. We use the prescription of [3] (slightly modified to take care of decoupled composite fields [17]) to compute the central charges. We do not find any violations of the bound (1.2); in fact in all of the interacting examples we consider the value of  $\alpha$  is negative and larger than  $-\frac{1}{2}$ .

We mostly follow notations and results from [17] and [16]. In particular, we will work in the large  $N$  limit

$$N_c \gg 1; \quad N_f \gg 1; \quad x \equiv \frac{N_c}{N_f} = \text{fixed}, \tag{1.4}$$

and study the quantity  $\alpha = a/c - 1$  as a function of the continuous parameter  $x$ .

The simplest non-trivial example is provided by supersymmetric  $QCD$ , whose field content consists of a vector superfield in the adjoint of  $SU(N_c)$ , and  $N_f$  chiral superfields  $Q$  and  $\tilde{Q}$  in the (anti)fundamental of  $SU(N_c)$ . The anomaly cancellation fixes the  $R$ -charges of the  $Q$  and  $\tilde{Q}$  to be

$$R(Q) = R(\tilde{Q}) = 1 - x. \tag{1.5}$$

The electric theory is asymptotically free for  $x > 1/3$ , while the magnetic theory (with the gauge group  $SU(N_f - N_c)$ ) is asymptotically free for  $x < 2/3$ . The two theories are Seiberg dual [18] (flow to the same IR fixed point). Using the relations (1.1) we compute the value of  $\alpha$ :

$$\alpha = \frac{1}{9x^2 - 7}, \tag{1.6}$$

which interpolates between  $-1/3$  and  $-1/6$  within the conformal window  $1/3 < x < 2/3$ , and thus satisfies the bounds (1.2). Note that the bound would naively be violated for e.g.

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<sup>1</sup>See [12] for related work.

$x > \sqrt{5}/3$ . However in this regime electric theory is strongly coupled and we should trust the magnetic description, which gives free theory in the IR.

One can discuss more general theories by adding adjoint chiral superfields. Asymptotic freedom implies that the number of chiral adjoints will be at most two. The rest of the paper is organized as follows. In sections 2 and 3 we consider supersymmetric  $QCD$  with one and two adjoints and possible relevant superpotentials, and briefly conclude in section 4.

## 2 $sQCD$ with one adjoint

### 2.1 Vanishing superpotential

The field content of supersymmetric  $QCD$  with one adjoint field consists of a vector and a chiral superfields both transforming in the adjoint of  $SU(N_c)$ , and  $N_f$  chiral superfields  $Q$  and  $\tilde{Q}$  in the (anti)fundamental of  $SU(N_c)$ . The theory is asymptotically free for  $x > 1/2$  (where  $x$  is defined in (1.4)), and we will restrict our discussion to this regime.

The anomaly cancellation condition implies that the  $R$ -charges of  $X, Q, \tilde{Q}$  depend on a single parameter. More precisely, denoting the  $R$ -charge of  $Q$  by  $y$ , gives the  $R$ -charge of  $X$  to be  $R(X) = (1 - y)/x$ . However, fixing the value of  $y$  is a nontrivial task. This problem has been solved by Intriligator and Wecht in [3], where using the techniques of superconformal calculus they proved that to determine the values of the  $R$ -charges one needs to maximize the trial  $a$ -function (given by (1.1)) with respect to  $y$ . In what follows we will discuss the scaled functions,

$$\tilde{a} = \frac{32}{3}a, \quad \tilde{c} = \frac{32}{3}c. \tag{2.1}$$

In the first approximation, i.e. without taking into account the decoupling of mesons which will be discussed shortly, the trial  $\tilde{a}$ -function is

$$\tilde{a}^{(0)}(x, y)/N_f^2 = 6x(y-1)^3 - 2x(y-1) + 3x^2 \left( \frac{1-y}{x} - 1 \right)^3 - x^2 \left( \frac{1-y}{x} - 1 \right) + 2x^2, \tag{2.2}$$

while the  $c$ -function is given by

$$\tilde{c}^{(0)}(x, y)/N_f^2 = 6x(y-1)^3 - \frac{10}{3}x(y-1) + 3x^2 \left( \frac{1-y}{x} - 1 \right)^3 - \frac{5}{3}x^2 \left( \frac{1-y}{x} - 1 \right) + \frac{4}{3}x^2, \tag{2.3}$$

and

$$\tilde{a}^{(0)} - \tilde{c}^{(0)} = \frac{2}{3}x(y-1). \tag{2.4}$$

Maximizing (2.2) with respect to  $y$  gives

$$R^{(0)}(Q) = y^{(0)} = \frac{3 + x(-3 - 6x + \sqrt{20x^2 - 1})}{3 - 6x^2}. \tag{2.5}$$

Plugging (2.5) into (2.2) and (2.3), one finds the following expression for  $\alpha$ :

$$\alpha(x) = \frac{4x^2 - 2}{13 - 44x^2 + 3\sqrt{20x^2 - 1}}, \tag{2.6}$$

which is a monotonic function interpolating between  $\alpha(1/2) = -1/8$  and  $\alpha(x \rightarrow \infty) = -1/11$ .

As explained in [17], this is not the full story as unitarity corrections due to decoupling mesons  $\mathcal{M}_j = \tilde{Q}X^{j-1}Q$  need to be taken into account. In particular, the naive  $R$ -charges of these superfields,

$$R(\mathcal{M}_j) = 2y + (j - 1)\frac{1 - y}{x}, \tag{2.7}$$

can get lower than the unitarity bound,  $R(\mathcal{M}_j) = \frac{2}{3}$ , as one changes the parameter  $x$ . In this situation the infrared  $CFT$  splits into an (in general) interacting theory and a decoupled free superfield  $\mathcal{M}_j$ . The correct central charges are obtained by subtracting the contribution of such mesons with the naive  $R$ -charges (2.7) and adding back the contribution due to the free fields, see [17] for details.

In practice, it is convenient to introduce central charges computed with the assumption that the first  $p(= 0, 1, 2, \dots)$  meson fields  $\mathcal{M}_1, \dots, \mathcal{M}_p$  are free:

$$\tilde{a}^{(p)} = \tilde{a}^{(0)} + \sum_{j=1}^p \left[ \frac{2}{9} - 3R(\mathcal{M}_j) + R(\mathcal{M}_j) \right], \tag{2.8}$$

and

$$\tilde{c}^{(p)} = \tilde{c}^{(0)} + \sum_{j=1}^p \left[ \frac{4}{9} - 3R(\mathcal{M}_j) + \frac{5}{3}R(\mathcal{M}_j) \right], \tag{2.9}$$

where  $\tilde{a}^{(0)}$  and  $\tilde{c}^{(0)}$  are given by (2.2) and (2.3) respectively. Substituting the values of  $R(\mathcal{M}_j)$  from (2.7) we get e.g.

$$\begin{aligned} \tilde{a}^{(p)}(x, y)/N_f^2 &= 6x(y - 1)^3 - 2x(y - 1) + 3x^2 \left( \frac{1 - y}{x} - 1 \right)^3 - x^2 \left( \frac{1 - y}{x} - 1 \right) + 2x^2 + \\ &\frac{1}{9} \sum_{j=1}^p \left[ 2 - 3 \left( 2y + (j - 1)\frac{1 - y}{x} \right) \right]^2 \left[ 5 - 3 \left( 2y + (j - 1)\frac{1 - y}{x} \right) \right]. \end{aligned} \tag{2.10}$$

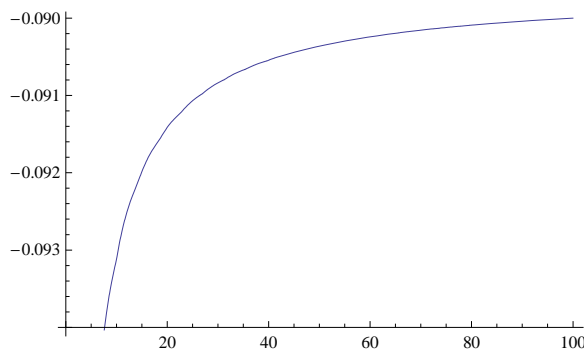
The central charges can then be determined via the following process: Start with  $\tilde{a}^{(0)}$  and find the value of  $x$  for which the  $R$ -charge of  $\mathcal{M}_1 = \tilde{Q}Q$  approaches  $2/3$ . At that point  $\mathcal{M}_1$  becomes free and we have to switch to the  $\tilde{a}^{(1)}$  description. Then look for the value of  $x$  at which  $\mathcal{M}_2$  becomes free and decouples, switch to  $\tilde{a}^{(2)}$ , etc. This process can be obviously continued to arbitrarily large  $x$ .

As shown in [17], this procedure is justified because once the meson decouples its  $R$ -charge (2.7) stays below the unitarity bound. We implemented this algorithm using Mathematica and computed the values of  $\tilde{a}(x)$ ,  $\tilde{c}(x)$ . The result for  $\alpha(x)$  is shown in figure 1.

## 2.2 Adjoint $sQCD$ with polynomial superpotential

Adding the relevant superpotential of the type,

$$W_k(X) = g_k \text{tr} X^{k+1}, \tag{2.11}$$



**Figure 1.**  $\alpha$  for  $sQCD$  with  $W = 0$ .

to adjoint  $sQCD$  induces the flow to a new fixed point which we call  $\mathbf{k}$  below. Of course, for a generic value of  $k$  the superpotential (2.11) is only relevant for sufficiently large  $x$ ,  $x > x_k$ . In [17]  $x_k$  was shown to be bounded from above,

$$x_k < \frac{4 - \sqrt{3}}{6}(k + 1). \tag{2.12}$$

This inequality is saturated in the limit  $k \rightarrow \infty$ . In the regime  $x > x_k$  the  $R$ -charges in the fixed point  $\mathbf{k}$  are fixed by the condition

$$R(X)(k + 1) = 2. \tag{2.13}$$

Hence, the central charges can be immediately computed and no a-maximization is required. As before, one needs to take care of the mesons whose  $R$ -charges

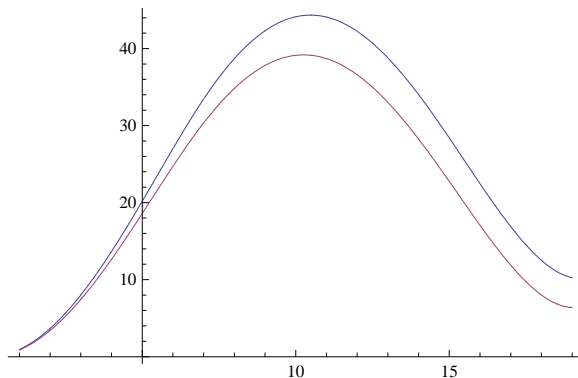
$$R(\mathcal{M}_j) = 2y_k + (j - 1)\frac{1 - y_k}{x} = 2\frac{j + k - 2x}{k + 1}. \tag{2.14}$$

violate the unitarity bound. Namely, we need to use (2.8) and (2.9) together with (2.13) and (2.14) to compute the central charges. As an example, we quote the result for  $\tilde{a}$ :

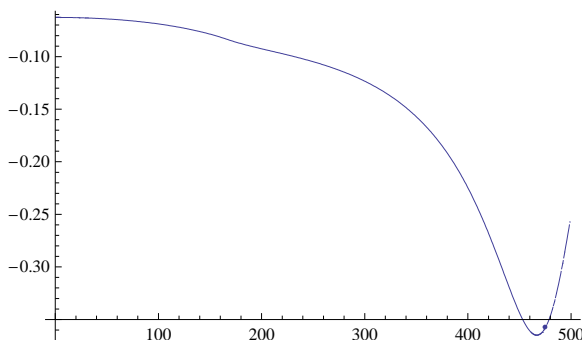
$$\begin{aligned} \tilde{a}_k(x)/N_f^2 &= 6x(y_k - 1)^3 - 2x(y_k - 1) + 3x^2 \left( \frac{1 - y_k}{x} - 1 \right)^3 - x^2 \left( \frac{1 - y_k}{x} - 1 \right) + 2x^2 + \\ &\quad \frac{1}{9} \sum_{j=1}^{p(x)} \left[ 2 - 3 \left( 2y_k + (j - 1)\frac{1 - y_k}{x} \right) \right]^2 \left[ 5 - 3 \left( 2y_k + (j - 1)\frac{1 - y_k}{x} \right) \right] = \\ &\quad \frac{4}{(k + 1)^3} \left[ x^2(2 + k + 5k^2 - 12x^2) - \frac{1}{9} \sum_{j=1}^{p(x)} (-5 + 6j + k - 12x)(1 - 3j - 2k + 6x)^2 \right]. \end{aligned} \tag{2.15}$$

Here  $p(x)$  is the number of mesons which are free at  $x$ ,

$$p(x) = \left\{ \begin{array}{ll} \lceil \frac{1}{3}(6x - 2k + 1) \rceil & \text{if } \lceil \frac{1}{3}(6x - 2k + 1) \rceil \leq k \\ k & \text{otherwise} \end{array} \right\}, \tag{2.16}$$



**Figure 2.**  $a(x)$  (red, bottom) and  $c(x)$  (blue, top) for  $sQCD$  with  $W = \text{tr}X^{21}$ .



**Figure 3.**  $\alpha(x)$  for  $sQCD$  with  $W = \text{tr}X^{501}$ .

where  $[\dots]$  is the integer part of the expression in brackets if the expression is positive and 0 otherwise. The  $p$ -th meson becomes free at

$$x(p) = \frac{1}{6}(3p + 2k - 1) . \tag{2.17}$$

Taking into account the mesons corrections is important for the positivity of  $a$  and  $c$ . In figure 2 these central charges are shown as functions of  $x$ . In figure 3  $\alpha$  is shown as a function of  $x$  for  $k = 500$ . Of course, these curves are only valid for  $x > x_k$  and, in addition, the region of validity is bounded from above. However it is already clear that the bounds on  $\alpha$  are not violated.

### 2.3 Strong-weak coupling duality

In this section, we will discuss the fixed point  $\mathbf{k}$  obtained by perturbing adjoint  $sQCD$  by the superpotential (2.11), where a dual description is known to exist [19–21], and one can ask what it predicts for the properties of the fixed point  $\mathbf{k}$  at strong coupling. The duality of [19–21] relates adjoint  $sQCD$  with gauge group  $SU(N_c)$  and superpotential (2.11) to an  $\mathcal{N} = 1$  supersymmetric gauge theory with gauge group  $SU(\tilde{N}_c) = SU(kN_f - N_c)$  and the following matter content: an adjoint field  $Y$ ,  $N_f$  chiral superfields  $q_i, \tilde{q}^i$  in the anti-fundamental and fundamental representation of the gauge group, respectively, and gauge



singlets  $(M_j)^i_j$ ,  $j = 1, \dots, k-1$ . The superpotential of this theory is given by

$$W_{\text{mag}} = -g'_k \text{tr} Y^{k+1} + \frac{1}{\mu^2} g'_k \sum_{j=1}^k M_j \tilde{q} Y^{k-j} q, \quad (2.18)$$

where  $\mu$  is an auxiliary scale. We will refer to these two theories as electric and magnetic, respectively. The conjecture of [19–21] is that they flow in the infrared to the same fixed point.

It will be convenient to introduce the magnetic dual of  $x$ ,

$$\tilde{x} \equiv \frac{\tilde{N}_c}{N_f} = k - x. \quad (2.19)$$

We will mainly discuss the region

$$x, \tilde{x} \in \left( \frac{1}{2}, k - \frac{1}{2} \right), \quad (2.20)$$

in which both the electric and the magnetic theories are asymptotically free.

When  $\tilde{x}$  is close to (and above)  $1/2$ , most of the terms in the superpotential (2.18) are irrelevant. The only exception is the term  $M_k \tilde{q} q$ , which is relevant in the infrared fixed point of the magnetic adjoint  $sQCD$  for all  $\tilde{x} > 1/2$ . This term drives the theory to a new fixed point, at which  $M_1, \dots, M_{k-1}$  are free but  $M_k$  is interacting. The  $R$ -charges and central charge  $\tilde{a}^m$  at this fixed point can be determined in a similar way to that employed in section 2. Denoting the  $R$ -charge of  $q, \tilde{q}$  by  $\tilde{y}$ , one has  $R(Y) = (1 - \tilde{y})/\tilde{x}$  and  $R(M_k) = 2 - 2\tilde{y}$ . Plugging these charges into the expression for  $\tilde{a}$ , and maximizing w.r.t.  $\tilde{y}$ , one can determine  $\tilde{y}$  and  $\tilde{a}^m$ . When  $\tilde{x}$  increases further, more and more of the terms in the superpotential (2.18) become relevant and have to be taken into account. As shown in [17], as long as  $\tilde{x} < \tilde{x}_k$ , one can proceed in a way similar to that employed in the previous section. The magnetic central charges computed with the assumption that the last  $p = (0, 1, 2, \dots)$  meson fields  $M_k, \dots, M_{k-p+1}$  are *not* free are given by the expressions similar to (2.8) and (2.9):

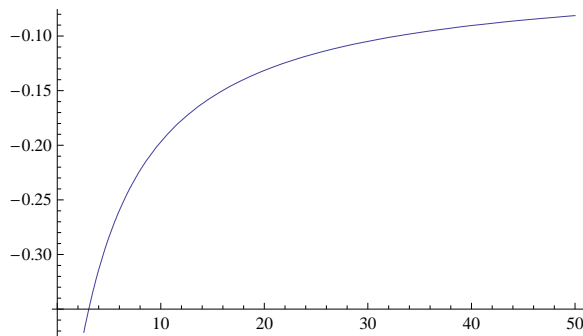
$$\tilde{a}^{m,(p)} = \tilde{a}^{m,(0)} + \sum_{j=1}^p \left[ 3R(M_j) + R(M_j) - \frac{2}{9} \right] + \frac{2}{9}k \quad (2.21)$$

and

$$\tilde{c}^{m,(p)} = \tilde{c}^{m,(0)} + \sum_{j=1}^p \left[ 3R(M_j) + \frac{5}{3}R(M_j) - \frac{4}{9} \right] + \frac{4}{9}k, \quad (2.22)$$

where  $\tilde{a}^{m,(0)}$  and  $\tilde{c}^{m,(0)}$  are given by (2.2) and (2.3) with the substitution  $x \rightarrow \tilde{x}$  and  $y \rightarrow \tilde{y}$ . The result for  $\tilde{a}^{m,(p)}$  reads

$$\begin{aligned} \tilde{a}^{m,(p)}/N_f^2 &= 6\tilde{x}(\tilde{y}-1)^3 - 2\tilde{x}(\tilde{y}-1) + 3\tilde{x}^2 \left( \frac{1-\tilde{y}}{\tilde{x}} - 1 \right)^3 - \tilde{x}^2 \left( \frac{1-\tilde{y}}{\tilde{x}} - 1 \right) + 2\tilde{x}^2 + \\ &\frac{1}{9} \sum_{j=1}^p \left[ 2 - 3 \left( 2\tilde{y} + (j-1) \frac{1-\tilde{y}}{\tilde{x}} \right) \right]^2 \left[ 5 - 3 \left( 2\tilde{y} + (j-1) \frac{1-\tilde{y}}{\tilde{x}} \right) \right] + \frac{2}{9}(k-2p). \end{aligned} \quad (2.23)$$



**Figure 4.**  $\alpha(\tilde{x})$  for magnetic  $sQCD$  at  $k = 50$ .

The  $R$ -charges can now be determined as follows. Start with  $\tilde{a}^{m,(1)}$  and maximize it w.r.t.  $\tilde{y}$ . Denote the value of  $\tilde{y}$  at the maximum by  $\tilde{y}^{(1)}(x)$ . Vary  $\tilde{x}$  to the point where the  $R$ -charge of  $M_{k-1}$  approaches  $2/3$ . At that point the term  $M_{k-1}\tilde{q}Yq$  in the magnetic superpotential (2.18) becomes relevant, and one should switch to the  $\tilde{a}^{m,(2)}$  description. This can be continued to arbitrarily large  $\tilde{x}$ .

The values of  $a$ ,  $c$  and  $\alpha$  can then be easily computed. The result for the latter is shown in figure 4. Apparently  $\alpha$  is closest to the lower bound when the magnetic theory is free, at  $\tilde{x} = 1/2$ .

It is interesting to analyze the theory near this point, where only  $M_k$  is not free. In this case, maximizing (2.23) with respect to  $y$  one arrives at

$$\tilde{y} = \frac{6\tilde{x}^2 - 9\tilde{x} - 3 - \sqrt{20\tilde{x}^4 - 48\tilde{x}^3 + 87\tilde{x}^2 - 16\tilde{x}}}{3(2\tilde{x}^2 - 8\tilde{x} - 1)}. \quad (2.24)$$

One can use this result to compute  $c$  and  $\alpha$  at  $\tilde{x} = 1/2$ :

$$\alpha\left(\tilde{x} = \frac{1}{2}\right) = -\frac{1}{2} + \frac{3}{4(k+2)}. \quad (2.25)$$

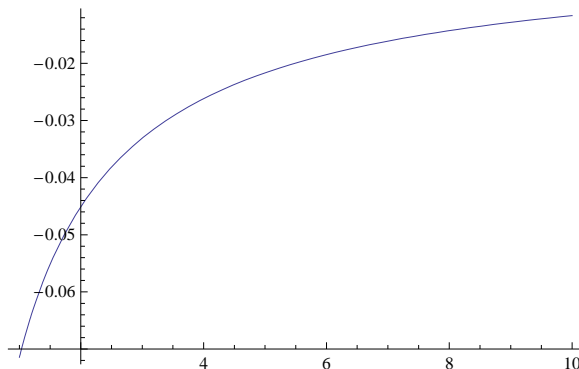
The lower bound is saturated in the limit  $k \rightarrow \infty$ . This is because in this limit there are infinitely many free chiral superfields which saturate the lower bound in (1.2).

### 3 $sQCD$ with two adjoints

We turn our attention to  $\mathcal{N} = 1$   $SYM$  theories with two adjoint chiral multiplets, which will be denoted by  $X$  and  $Y$ . Asymptotic freedom implies here that  $x \geq 1$ . Computing the  $R$  charges in the  $IR$  for these theories was extensively discussed in [16] and we will closely follow this analysis.

The simplest theory (denoted by  $\hat{O}$  in [16]) is the theory with two adjoints without a superpotential. One can find the  $R$ -charges by  $a$ -maximization. The results are

$$\begin{aligned} R(Q) = R(\tilde{Q}) &= 1 + \frac{3x - 2x\sqrt{26x^2 - 1}}{3(8x^2 - 1)}, \\ R(X) = R(Y) &= \frac{1}{2} + \frac{-3 + 2\sqrt{26x^2 - 1}}{6(8x^2 - 1)}. \end{aligned} \quad (3.1)$$



**Figure 5.**  $\alpha(x)$  for the  $\hat{O}$  theory.

There are no gauge invariant composites which violate the unitarity bound for any value of  $x$  as all the  $R$  charges are larger than half. In figure 5 we depict  $\alpha(x)$  for this theory.

The nontrivial  $IR$  fixed points of the  $\mathcal{N} = 1$   $SYM$  theories with two adjoints and a superpotential are classified using an  $ADE$ -like structure [16]. The  $A$  series is equivalent in the  $IR$  to the theory with single adjoint discussed in the previous section, and we will discuss in what follows the  $D$  and the  $E$  series.

### 3.1 $\hat{D}$

We start our discussion with the  $D$  series. The superpotential is given by

$$W_{D_{k+2}} = Tr \left( X^{k+1} + XY^2 \right). \quad (3.2)$$

The first theory in this sequence is obtained by taking  $k = -1$  above, and is denoted as  $\hat{D}$ . One can choose the following parameterization for the  $R$  charges

$$R(Q) = R(\tilde{Q}) = y, \quad R(Y) = \frac{y-1}{x} + 1, \quad R(X) = \frac{2-2y}{x}. \quad (3.3)$$

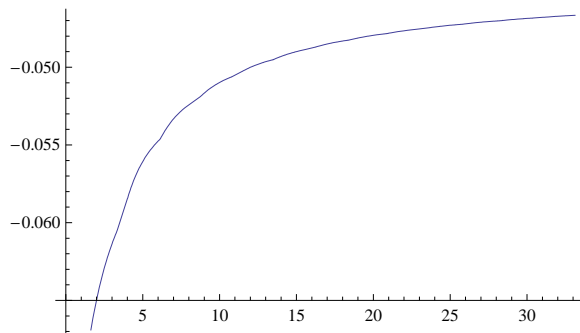
To determine the  $R$  charges there is a need for  $a$ -maximization. The trial functions thus take the following form,

$$\begin{aligned} \tilde{a}_{\hat{D}}^{(0)}/N_f^2 &= 2x^2 + 3x^2 \left( \frac{y-1}{x} \right)^3 + 3x^2 \left( \frac{2-2y}{x} \right)^3 + 6x(y-1)^3 \\ &\quad - x^2 \left( \frac{y-1}{x} \right) - x^2 \left( \frac{2-2y}{x} \right) - 2x(y-1). \end{aligned} \quad (3.4)$$

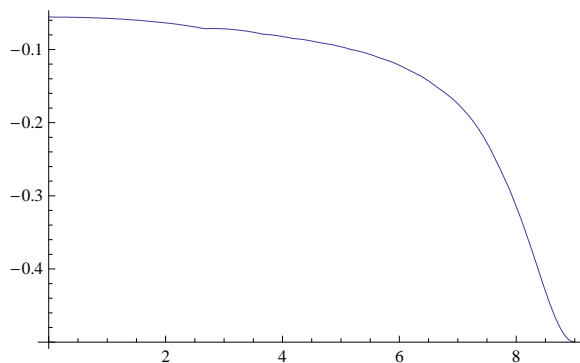
$$\begin{aligned} \tilde{c}_{\hat{D}}^{(0)}/N_f^2 &= \frac{4}{3}x^2 + 3x^2 \left( \frac{y-1}{x} \right)^3 + 3x^2 \left( \frac{2-2y}{x} \right)^3 + 6x(y-1)^3 \\ &\quad - \frac{5}{3}x^2 \left( \frac{y-1}{x} \right) - \frac{5}{3}x^2 \left( \frac{2-2y}{x} \right) - \frac{10}{3}x(y-1). \end{aligned} \quad (3.5)$$

The maximization of  $\tilde{a}_{\hat{D}}$  gives

$$y^{(0)} = 1 + \frac{x(12 - \sqrt{11 + 38x^2})}{3(2x^2 - 7)}. \quad (3.6)$$



**Figure 6.**  $\alpha(x)$  for the  $\hat{D}$  theory.



**Figure 7.**  $\alpha(x)$  for the  $D_5$  theory,  $k = 3$ .

We have to take into account the decoupling mesons. The mesons in this theory are  $\tilde{Q}X^lY^jQ$ , where  $j = 0, 1$  and  $l$  is non-negative. From (3.6) and (3.3) the only mesons which can violate the unitarity bound have  $j = 0$ . No baryons ever hit the unitarity bound in the  $\hat{D}$  theory [16]. The  $\alpha(x)$  function for the  $\hat{D}$  theory is depicted on figure 6.

### 3.2 $D_{k+2}$

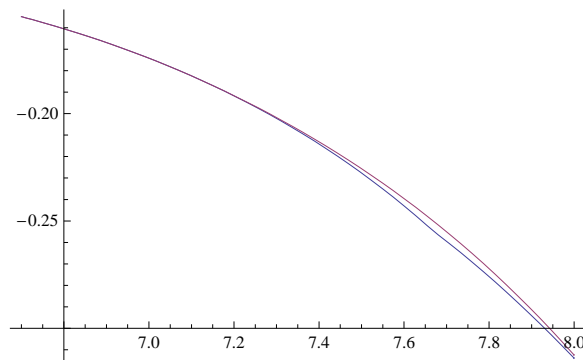
For  $k > -1$  there is no need for  $a$ -maximization as the charges are fixed by the marginality of the superpotential in the  $IR$ . To actually compute  $a$  we still have to take into account the decoupled free composites. The case of even and odd  $k$  are qualitatively different. The chiral ring relations are

$$\{X, Y\} = 0, \quad X^k + Y^2 = 0. \tag{3.7}$$

For  $k$  odd these imply that  $Y^3 = 0$  and the spectrum of mesons is truncated. For even  $k$  there is no such truncation. First, we concentrate on the odd case.

There are no decoupling baryons in the stability region  $x < 3k$  [16], and thus only the mesons have to be taken into account. The results are depicted in figure 7 for the  $\hat{D}_5$  theory (for all odd  $k$  the results are qualitatively similar). The lower bound on the ratio of the central charges is saturated exactly at  $x = 3k$  for any odd  $k$ .

However, as we approach  $x = 3k$  one should switch to the magnetic description of the model as it provides us with a more reliable picture of the physics. The magnetic dual of



**Figure 8.**  $\alpha(x)$  for the magnetic (top, red) and electric (bottom, blue)  $D_5$  theories.

the  $D_{k+2}$  theories was introduced in [22]. This theory has an  $SU(3kN_f - N_c)$  gauge group, two adjoints  $\tilde{X}$  and  $\tilde{Y}$ ,  $N_f$  quarks  $q_i$  and  $\tilde{q}_i$ , and  $3kN_f$  gauge singlets  $M_{lj}$  ( $l = 1, \dots, 3$ ,  $k = 1, 2, 3$ ). The tree level superpotential is

$$W = a_0 \text{Tr} \tilde{X}^{k+1} + a_1 \text{Tr} \tilde{X} \tilde{Y}^2 + \sum_{k,l} b_{lj} M_{lj} \tilde{q} \tilde{X}^{k-l} \tilde{Y}^{3-j} q. \quad (3.8)$$

One can compute the  $a$  and  $c$  charges in the magnetic theory using the techniques applied to the electric theory above [16].

Depending on the value of  $x$  we should trust one of the results, magnetic or electric. The explicit picture is as follows. For  $x \leq 1$  the theory is free in  $IR$ . Beyond  $x = 1$  it becomes asymptotically free. For  $x < x_{D_{k+2}}^{\min}$  the interaction  $\text{Tr} X^{k+1}$  is irrelevant and the theory flows to the  $\hat{D}$  point. By definition at  $x = x_{D_{k+2}}^{\min}$  the interaction becomes relevant and the electric theory (3.2) gives the correct physics. For magnetic theory it is useful to define  $\tilde{x} = 3k - x$ , and the analysis above can be repeated with the magnetic dual and by exchanging  $x$  with  $\tilde{x}$ . For  $\tilde{x} < \tilde{x}_{D_{k+2}}^{\min}$  the magnetic theory flows to a “magnetic” version of the  $\hat{D}$  point. In the conformal window,  $x_{D_{k+2}}^{\min} < x < 3k - \tilde{x}_{D_{k+2}}^{\min}$ , both descriptions should agree. In figure 8 we depict the behavior of  $\alpha(x)$  near  $\tilde{x} = \tilde{x}_{D_{k+2}}^{\min}$  for  $k = 3$  in the magnetic theory. The conformal window begins at  $\tilde{x}_{D_5}^{\min} \sim 1.86$  ( $x \sim 7.14$ ) and for  $x < \tilde{x}_{D_5}^{\min}$  the electric result depicted in figure 7 can be trusted.

One can repeat the analysis above for the case of even  $k$ . In figures 9 and 10 we depict the results for  $k = 2$  and  $k = 4$ .

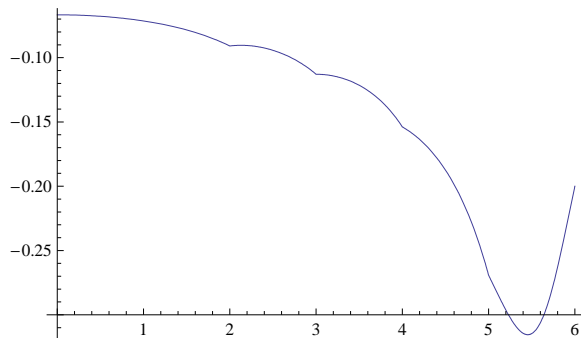
### 3.3 $\hat{E}$

Next we turn our attention to the  $E$  series. The  $E$  series is defined by the superpotential  $W_{\hat{E}} = \text{Tr} Y^3$  and the following deformations,

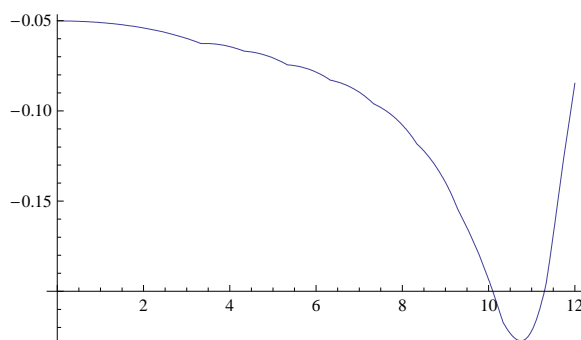
$$W_{E_6} = \text{Tr} (X^4 + Y^3), \quad W_{E_7} = \text{Tr} (Y X^3 + Y^3), \quad W_{E_8} = \text{Tr} (X^5 + Y^3). \quad (3.9)$$

We start by considering the basic case  $W_{\hat{E}}$ . The  $R$ -charges are parameterized as

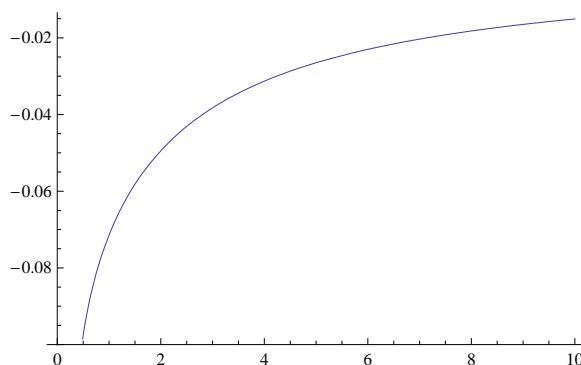
$$R(Y) = \frac{2}{3}, \quad R(Q) = R(\tilde{Q}) = y, \quad R(X) = \frac{1+x-y}{x} - \frac{2}{3}. \quad (3.10)$$



**Figure 9.**  $\alpha(x)$  for the  $D_2$  theory.



**Figure 10.**  $\alpha(x)$  for the  $D_4$  theory.



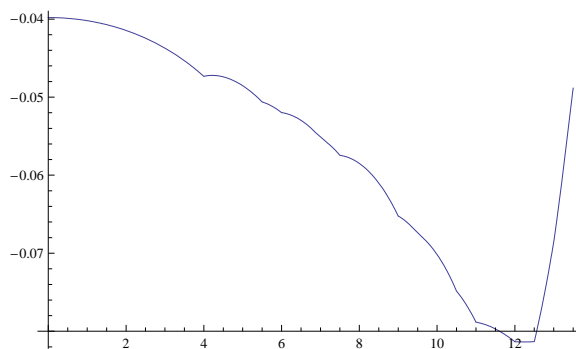
**Figure 11.**  $\alpha(x)$  for the  $\hat{E}$  theory.

As we have a free parameter  $y$  we have to use  $a$ -maximization. Here we do not have any gauge invariant operators violating the unitarity bound and thus the calculation is straightforward. In figure 11 we depict the relevant diagram.

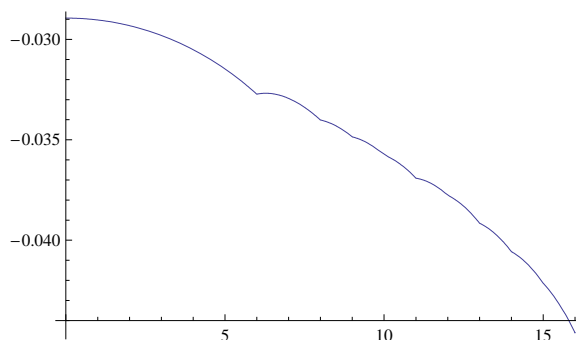
### 3.4 $E_6$

There are three interesting deformations of the  $\hat{E}$  theory. We start with the  $E_6$  point. Here the  $R$ -charges are fixed by demanding marginality of the superpotential,

$$R(Y) = \frac{2}{3}, \quad R(Q) = R(\tilde{Q}) = 1 - \frac{x}{6}, \quad R(X) = \frac{1}{2}. \quad (3.11)$$



**Figure 12.**  $\alpha(x)$  for the  $E_6$  theory. The theory is conjectured to possess a quantum instability at roughly  $x = 13.8$ .



**Figure 13.**  $\alpha(x)$  for the  $E_7$  theory.

However we have here decoupling mesons (and baryons). A meson with  $k$   $Y$  fields and  $l$   $X$  fields decouples at

$$x = 4 + \frac{3}{2}l + 2k. \tag{3.12}$$

One has to account for the different mesons keeping in mind the chiral ring identifications  $Y^2 = 0$  and  $X^3 = 0$ . In figure 12 we depict the results.

### 3.5 $E_7$

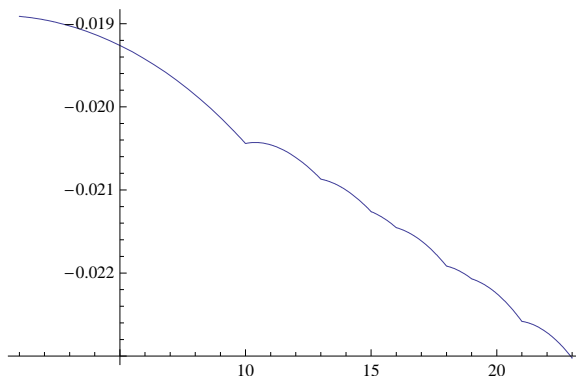
Let us discuss the  $E_7$  point. Here the  $R$ -charges are fixed as,

$$R(Y) = \frac{2}{3}, \quad R(Q) = R(\tilde{Q}) = 1 - \frac{x}{9}, \quad R(X) = \frac{4}{9}. \tag{3.13}$$

We have here decoupling mesons (and baryons). A meson with  $k$   $Y$  fields and  $l$   $X$  fields decouples at

$$x = 6 + 2l + 3k. \tag{3.14}$$

One has to account for the different mesons keeping in mind the chiral ring identifications  $\{Y, X^2\} = 0$  and  $X^3 + 3Y^2 = 0$ . In figure 13 we depict the results.



**Figure 14.**  $\alpha(x)$  for the  $E_8$  theory.

### 3.6 $E_8$

Finally we discuss the  $E_8$  case. The superpotential has the form  $Tr(X^5 + Y^3)$ . This implies the chiral ring relations  $X^4 = 0$  and  $Y^2 = 0$ . The  $R$  charges are as follows

$$R(Q) = R(\tilde{Q}) = 1 - \frac{x}{15}, \quad R(X) = \frac{2}{5}, \quad R(Y) = \frac{2}{3}. \quad (3.15)$$

There is no  $a$ -maximization but we have to take into account decoupling mesons and baryons. The lowest lying mesons are summarized below,

$$\begin{aligned} & (\tilde{Q}Q, 10), \quad (\tilde{Q}XQ, 13), \quad (\tilde{Q}YQ, 15), \quad (\tilde{Q}X^2Q, 16), \quad (\tilde{Q}XYQ, \tilde{Q}YXQ, 18), \\ & (\tilde{Q}X^3Q, 19), \quad (\tilde{Q}YX^2Q, \tilde{Q}XYXQ, \tilde{Q}X^2YQ, 21), \quad (\tilde{Q}YXYQ, 23), \end{aligned} \quad (3.16)$$

where we also included the value of  $x$  for which the mesons decouple. The baryons are built from the dressed quarks

$$\hat{Q}_{\beta,\alpha,\gamma}^{(n)} = X^\beta \left[ \prod_{i=1}^n YX^{\alpha_i} \right] Y^\gamma Q, \quad \alpha_i = 1, 2, 3 \quad \gamma = 0, 1 \quad \beta = 0, 1, 2, 3. \quad (3.17)$$

Thus the baryons are products of  $N_c$  of these dressed quarks. For sufficiently large values of  $x$  the baryons will decouple and the proof is the same as for the  $E_6$  case [16]. We can find lower bounds on the values of  $x$  for the baryon decoupling,  $x_\star$ . The  $R$  charge of any baryon is greater than

$$N_c \left( 1 - \frac{x}{15} \right) + \frac{2}{5}(N_c - N_f), \quad (3.18)$$

just by taking  $N_f$  un-dressed quarks and the rest being dressed with  $X$ . Thus, we get a very rough lower bound of  $x_\star > 20$ .

## 4 Conclusions

In this note we have computed the value of  $\alpha = a/c - 1$  for a wide range of interacting  $\mathcal{N} = 1$  superconformal theories. We have verified that the bound (1.2) suggested by



Hofman and Maldacena in [11] is satisfied in these theories. It is interesting to note that in all the interacting cases considered  $\alpha$  is actually negative (and larger than  $-1/2$ ). It is also worth noting that for interacting theories with gravity duals, which presumably are strongly coupled,  $\alpha = 0$ . On the other hand, in our examples strong coupling regime, which would happen in the middle of the conformal window, was not associated with small values of  $\alpha$ . As far as outlook for the future is concerned, it would perhaps be interesting to consider more examples of *CFTs* in order to understand better the significance of  $\alpha$ . It would also be great to have a proof of (1.2) and of its non-supersymmetric cousin.

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